EXTENSION OF THE PRINCIPLE OF GAUSS TO SYSTEMS WITH DRY (COULOMB) FRICTION

(RASPROSTRANENIE PRINTSIPA GAUSSA NA SISTEMY S SUKHIM TRENIEM)

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In [1] Painlevé presented the foundations of a general theory of motion of mechanical systems with dry friction. He showed also that the hypotheses of the existence of solid bodies in conjunction with the proposition on the finiteness of the accelerations of the system can lead to the contradiction of the law of Coulomb friction.

A number of scientists, among them F. Klein, R. Mises, H. Hamel and L. Prandtl, have participated in the discussions of this paradox. As a result of these considerations, there have evolved several possible ways of overcoming this paradox by either rejecting one of the hypotheses or by changing the law of friction.

The equations, obtained by Painlevé for the determination of the accelerations of the system with friction, in terms of the initial conditions which correspond to the zero relative sliding velocities of the touching surfaces, were written in such a general form that certain observable peculiarities of the law of Coulomb friction still remain unexplained. These equations are formally applicable, for example, to systems with "servoconnections" considered by Béguin [2]. By considering initial conditions which assign zero values to some of the relative sliding velocities, Painlevé writes: "The investigation of the case of static friction is quite complicated but it is impossible to simplify it", and from then on he restricts his investigation to the consideration of special cases. In his book there are given, however, the analyses of the motions of such a large number of special mechanical systems, that all concrete rules for the construction of the equations are clearly displayed. Therefore, one does not encounter any great difficulties in the derivation of the general equations of motion in the form which applies specifically and exclusively to the law of Coulomb friction. The first part of the present work is devoted to this problem.

In the second part of this work there are considered systems in which all the maxima of the frictional forces can be determined in terms of the coordinates, the velocities, time, and of the active forces, up to the determination of the accelerations of the system. It is shown that the actual acceleration of the system attains a minimum of a certain function depending on the acceleration, and differing from the Gaussian constraint only by a term which contains the maxima of the frictional forces. This variational principle makes it possible to isolate the accelerations that correspond to the law of friction even for initial conditions which involve zero values of some of the relative velocities of sliding.

1. Let us consider a mechanical system, with holonomic coordinates $q_1, \ldots, q_{n+k+l}$, subjected to ideal holonomic constraints by means of nonholonomic linear relations

$$A_{i1}\dot{q}_1 + \ldots + A_{i,n+k+l}\dot{q}_{n+k+l} = 0 \qquad (i = 1,\ldots,l) \tag{1.1}$$

with the determination of the possible displacements

$$A_{i1}\delta q_1 + \ldots + A_{i,n+k+l}\delta q_{n+k+l} = 0 \qquad (i = 1,\ldots,l) \tag{1.2}$$

and the holonomic free contacts with Coulomb friction

$$q_{n+1} \leqslant 0, \ldots, q_{n+k} \leqslant 0 \tag{1.3}$$

which express the fact that if the relations (1.3) are equalities, then the points or bodies of the system slide with friction over the bodies of the system or over bodies which are external to the system.

For the sake of simplicity, we assume that all constraints do not depend on time explicitly.

The law of Coulomb friction is expressed in the following form.

Let the body S_1 (or point) of the system be in contact with the body S_2 at the point p in space. We assume that there exists a normal either to the body S_1 or to the body S_2 at the point of contact p. If $N > 0$ is the normal force of pressure of the body S_2 on S_1 directed inward to S_1, while $\mathbf{v}$ is the velocity of the point p_1 of S_1 which is in contact with the body S_2, then the frictional force is equal to kN applied at the point p_1 and directed in the opposite direction to $\mathbf{v}$, where $k > 0$ is the coefficient of friction and depends only on the coordinates of the point of contact of the bodies S_1 and S_2.

Disregarding for the time being the case when $\mathbf{v} = 0$, let us consider the state of the system

$$q_1, \ldots, q_n, q_{n+1}=0, \ldots, q_{n+k}=0, \; q_{n+k+1}, \ldots, \; q_{n+k+l}$$
$$\dot{q}_1, \ldots, \dot{q}_n, \dot{q}_{n+1}=0, \ldots, \; \dot{q}_{n+k}=0, \; \dot{q}_{n+k+1}, \ldots, \; \dot{q}_{n+k+l}$$

which corresponds to (1.2) and which is such that not a single one of the relative velocities is zero.

Let us impart to the system a possible displacement

$$\delta q_1, \ldots, \delta q_n, \; \delta q_{n+1} \leqslant 0, \ldots, \delta q_{n+k} \leqslant 0 \qquad (1.4)$$

From here on we shall assume that the first $n + k$ displacements have been selected as the independent displacements.

Let us consider all the space points $p_1, \ldots, p_s$ at which there occurs a contact. Since by hypothesis at each of these points there exists a normal to at least one of the bodies, we can select a coordinate system whose z-axis is directed along the outer normal, while the x_i- and y_i-axes are at right angles to each other and to the z-axis, and are fixed in the body.

Suppose that in consequence of the displacement (1.4) every point of the bodies which is at the ith contact point has received, relative to the ith system of coordinates, the displacements δx_i, δy_i, $\delta z_i (i = 1, \ldots, s)$.

In this manner we can number all displacements, because only one of two contact points can have a nonzero relative displacement.

Since δz_i vanishes when $\delta_{n+1} = \ldots = \delta_{n+k} = 0$, the expressions δx_i, δy_i, δz_i will have the following form in terms of the δq_i:

$$\delta x_i = \alpha_{i1}^1 \delta q_1 + \ldots + \alpha_{i, n+k}^1 \delta q_{n+k}$$
$$\delta y_i = \alpha_{i1}^2 \delta q_1 + \ldots + \alpha_{i, n+k}^2 \delta q_{n+k}$$
$$\delta z_i = \alpha_{i, n+1}^3 \delta q_{n+1} + \ldots + \alpha_{i, n+k}^3 \delta q_{n+k}$$
$$\alpha_{i1}^3 = \ldots = \alpha_{in}^3 = 0 \qquad (1.5)$$

The work of the normal reaction and of the friction on the virtual displacements δx_i, δy_i, δz_i is equal to

$$-k_i N_i \frac{v_{ix}}{|\mathbf{v}_i|} \delta x_i - k_i N_i \frac{v_{iy}}{|\mathbf{v}_i|} \delta y_i + N_i \delta z_i$$

Here $\mathbf{v}_i$ is the relative velocity with the projections v_{ix}, v_{iy} on the x_i- and y_i-axes. Setting in turn all of the displacements $\delta q_i, \ldots, \delta q_{n+k}$

equal to zero except the δq_j displacement, we obtain the equation

$$\frac{\partial S}{\partial \ddot{q}_j} = Q_j + \sum_{i=1}^{s} - k_i N_i \frac{v_{ix}\alpha_{ij}^1 + v_{iy}\alpha_{ij}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}} + N_i \alpha_{ij}^3 \qquad (j = 1, \ldots, n+k) \quad (1.6)$$

where S is the energy of acceleration for the system without constraints (1.3), and Q_j are generalized forces.

If

$$2S = \sum_{ij=1}^{n+k} \gamma_{ij}\ddot{q}_i\ddot{q}_j + \beta_i \ddot{q}_i + \delta$$

where γ_{ij}, β_i, δ do not depend on $\ddot{q}_j$, then the equation

$$u_j = \frac{\partial S}{\partial \ddot{q}_j} - \beta_j = \gamma_{1j}\ddot{q}_1 + \ldots + \gamma_{n+k,\, j}\ddot{q}_{n+k} \qquad (j = 1, \ldots, n+k)$$

will have a unique solution since the quadratic form

$$\sum_{i,\, j=1}^{n+k} \gamma_{ij}\ddot{q}_i\ddot{q}_j$$

is positive-definite and its discriminant which coincides with the determinant of the last system is not equal to zero. Let its solution be

$$\ddot{q}_j = \gamma_{1j}' u_1 + \ldots + \gamma_{n+k,\, j}' u_{n+k}$$

Solving the system (1.6), we obtain

$$\ddot{q}_j = \sum_{m=1}^{n+k} \gamma_{mj}' \left(Q_m - \beta_m + \sum_{i=1}^{s} - k_i N_i \frac{v_{ix}\alpha_{im}^1 + v_{iy}\alpha_{im}^2}{|\mathbf{v}_i|} + \alpha_{im}^3 N_i \right)$$

$$(j = 1, \ldots, n+k)$$

Setting all $\ddot{q}_{n+1}, \ldots, \ddot{q}_{n+k}$ equal to zero, we obtain the equations for the determination of the reactions

$$\sum_{m=1}^{n+k} \gamma_{mj}' \left(Q_m - \beta_m + \sum_{i=1}^{s} - k_i N_i \frac{v_{ix}\alpha_{im}^1 + v_{iy}\alpha_{im}^2}{|\mathbf{v}_i|} + \alpha_{im}^3 N_i \right) = 0$$

$$(j = n+1, \ldots, n+k)$$

If it is possible by means of these equations to determine the values of all the linear combinations N_i on which the right-hand sides of the first n equations depend, and if all these linear combinations can be satisfied with some positive N_i, then the motion will be determined in a unique way. We shall illustrate this last assertion with an example. Suppose that a solid body can be subjected to a plane sliding motion along a fixed rough path along the x-axis, on which it rests over a segment AB. If the coefficient of friction k is constant of all points on the x-axis, then the frictional force will not depend on the distribution of the normal pressures on the segment AB, but it will depend only on their geometrical sum and will be equal to $+kY\dot{x}\ |\dot{x}|$, where Y is the sum of the projections of all the forces applied to the body upon the y-axis perpendicular to the x-axis, and $\dot{x}$ is the velocity of the body. This shows that under nonzero initial velocity conditions the motion is determined uniquely, even if one cannot determine the distribution of the normal pressure.

2. The consideration of the case of zero initial velocity will begin with an example.

Let us consider a light rod carrying point masses m_1 and m_2 at the distance a from each other. Suppose that these masses press, upon a rough plane surface with a coefficient of friction k, with forces $\mathbf{N}_1$, and $\mathbf{N}_2$ normal to the surface. Furthermore, let us suppose that a force $\mathbf{F}$ is applied in this plane to the rod at a distance b from the point m_1 in the direction from b_1 to b_2, and forming an angle ϕ with the direction from m_1 to m_2.

Depending on the parameters of the problem, four cases can occur: (1) the rod remains at rest; (2) the rod rotates around m_1; (3) the rod rotates around m_2; (4) the rod rotates around a point O_1, distinct from m_1 and m_2, or it undergoes a translation.

All the terms "rest", "rotation" and "translation" should be understood in the sense of the distribution of the accelerations of the initial instant.

In accordance with the law of friction, the state of rest will occur if the equations of equilibrium for the rod

$$aR_1 \sin\alpha_1 + (a - b) F \sin\varphi = 0 \tag{2.1}$$

$$aR_2 \sin\alpha_2 + bF \sin\varphi = 0 \tag{2.2}$$

$$R_1 \cos\alpha_1 + R_2 \cos\alpha_2 + F \cos\varphi = 0 \tag{2.3}$$

can be satisfied by reactions $|R_1| \leqslant kN_1$, $|R_2| < kN_2$. Here, a_1 and a_2 are angles formed by the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ with the x-axis, directed from m_1 to m_2. The first two equations are the equations of the moments relative to m_2 and m_1, while the third equation represents the projection upon the x-axis.

The rotation of the rod around m_1 with the angular acceleration ϵ will occur if in the presence of the force kN_2 and the force of inertia m_2a, directed against the acceleration of the point m_2, the equilibrium equations

$$a\ R_1 \sin\alpha_1 + (a - b) F \sin\varphi = 0 \tag{2.4}$$

$$a(kN_2 + m_2\varepsilon a) = bF\sin\varphi \tag{2.5}$$

$$R_1\cos\alpha_1 + F\cos\varphi = 0 \tag{2.6}$$

can be satisfied by a reaction $|R_1| < kN_1$.

Rotation around m_2 will take place if the equations

$$(kN_1 + m_1\varepsilon a)\, a = (a - b) F \sin\varphi \tag{2.7}$$

$$aR_2\sin\alpha_2 + bF\sin\varphi = 0 \tag{2.8}$$

$$R_2\cos\alpha_2 + F\cos\varphi = 0 \tag{2.9}$$

can be satisfied by same ϵ and by an $|R_2| < kN_2$.

If we denote by r_1 and r_2 the distances of the points m_1 and m_2 from the center O_1, by h the distance of the line of action of the force $\mathbf{F}$ from O_1, and by $(kN_1 + m_1\epsilon r_1)_K$ the projection on x of the vector $(kN_1 + m_1\epsilon r_1)$ directed against the acceleration of m_1, and so on, then the rotation around O_1 will satisfy the equations

$$(kN_1 + m_1\varepsilon r_1)\, r_1 + (kN_2 + m_2\varepsilon r_2)\, r_2 = Fh \tag{2.10}$$

$$(kN_1 + m_1\varepsilon r_1)_x + (kN_2 + m_2\varepsilon r_2)_x + F\cos\varphi = 0 \tag{2.11}$$

$$(kN_1 + m_1\varepsilon r_1)_y + (kN_2 + m_2\varepsilon r_2)_y + F\sin\varphi = 0 \tag{2.12}$$

Here, the first equation represents the moments equation about O_1, the second one is the equation of projections on the x-axis, where, in consequence of a known theorem in kinematics, the terms $(kN_1 + m_1\epsilon r_1)_x$ and $(kN_2 + m_2\epsilon r_2)_x$ have the same sign. The third equation is the equation of projections on the y-axis, at right angles to the x-axis and directed towards the force $\mathbf{F}$.

The translational displacement with acceleration w satisfies the equations

$$kN_1 + m_1 w + kN_2 + m_2 w = F \tag{2.13}$$

$$(kN_1 + m_1 w)\, a = (a - b) F \tag{2.14}$$

$$(kN_2 + m_2 w)\, a = Fb \tag{2.15}$$

Under such a large number of distinct possibilities there arise three questions: (1) Is it possible that for the same parameters there can occur several variants? (2) Is it possible that the equations for a given variant, for example the fourth one, can have more than one solution? (3) Do there exist parameters for which not one of these variants can occur?

Comparisons of Equations (2.2) with (2.5), of (2.1) with (2.7) and of (2.5) with (2.8) show that not more than one of the first three variants can occur.

Since the moments of the forces R_1 and R_2 relative to O_1 cannot be less in absolute value than $(kN_1 + m_1\epsilon r_1)r_1$ and $(kN_2 + m_2\epsilon r_2)r_2$, respectively, and since (2.1) contradicts (2.14), it follows that the first and fourth variant also exclude each other.

As a consequence of Equations (2.10), (2.11), (2.12) we have the equation of the moments relative to the point m_2:

$$(kN_1 + m_1\varepsilon r_1)_y a + (a - b) \sin\varphi = 0 F$$

Comparing this equation with (2.4), we obtain

$$(kN_1 + m_1\varepsilon r_1)_y = R_1 \sin\alpha_1$$

From (2.11) and (2.6) it follows that

$$(kN_1 + m_1\varepsilon r_1)_x + (kN_2 + m_2\varepsilon r_2)_x = R_1 \cos\alpha_1$$

As mentioned above, both terms on the left-hand side of the last equation have the same sign.

Squaring the terms of the last two equations and adding the results, we obtain

$$(kN_1 + m_2\varepsilon r_1)^2 + 2(kN_1 + m_1\varepsilon r_1)_x (kN_2 + m_2\varepsilon r_2)_x + \\ + (kN_2 + m_2\varepsilon r_2)_x{}^2 = R_1{}^2 > k^2N_1{}^2$$

which is impossible. Comparing (2.6) and (2.13), we conclude that the second and fourth variant exclude each other. An analogous proof can be given for the third and fourth variants.

Thus we have arrived at a negative answer to the first question. This means that for arbitrarily given parameters there can occur only one variant.

In the sequel it is convenient to denote by $\ddot{x}$ and $\ddot{y}$ the components of acceleration of the point m_1, and by $\ddot{x}$ and $\ddot{y} + \epsilon a$ the components of acceleration of the point m_2. We can now write the equations for the fourth variant in the form

$$(m_1 + m_2)\ddot{x} = F\cos\varphi - kN_1\frac{\ddot{x}}{\sqrt{\ddot{x}^2+\ddot{y}^2}} - kN_2\frac{\ddot{x}}{\sqrt{\ddot{x}^2+(\ddot{y}+\varepsilon a)^2}}$$

$$m_1\ddot{y} + m_2(\ddot{y}+\varepsilon a) = F\sin\varphi - kN_1\frac{\ddot{y}}{\sqrt{\ddot{x}^2+\ddot{y}^2}} - kN_2\frac{\ddot{y}+\varepsilon a}{\sqrt{\ddot{x}^2+(\ddot{y}+\varepsilon a)^2}}$$

$$m_2(\ddot{y}+\varepsilon a)a = Fb\sin\varphi - kN_2\frac{a(\ddot{y}+\varepsilon a)}{\sqrt{\ddot{x}^2+(\ddot{y}+\varepsilon a)^2}}$$

It is not difficult to notice that these equations are the equations for the extremum of the function

$$S + \Psi = \frac{1}{2}m_1(\ddot{x}^2+\ddot{y}^2) + \frac{1}{2}m_2(\ddot{x}^2+(\ddot{y}+\varepsilon a)^2) - F\ddot{x}\cos\varphi -$$
$$- F(\ddot{y}+b\varepsilon)\sin\varphi + kN_1\sqrt{\ddot{x}^2+\ddot{y}^2} + kN_2\sqrt{\ddot{x}^2+(\ddot{y}+\varepsilon a)^2}$$

which consists of the energy of acceleration S and of the function Ψ which is a homogeneous function of order one in terms of the relative accelerations. Below we shall show for the general case that:

a) The accelerations corresponding to any one of the four variants give an isolated minimum of the function $S + \Psi$;

b) The equations of any one of the variants admit only one solution for the relative accelerations (motion). (We shall say that to the first variant there corresponds the zero solution which is, obviously, a unique solution for this case);

c) The function $S + \Psi$ has at least one isolated minimum;

d) In order that the function $S + \Psi$ have an isolated minimum at some point of the space of possible accelerations, it is sufficient and necessary that the function π, which is homogeneous of the first order relative to the accelerations, (a part of the deviation of the function $S + \Psi$) take on only non-negative values.

In so far as (b) contains a negative answer to the second question, we need to give an answer only to the third question.

Let us suppose that the first variant is not fulfilled, i.e. the equations of equilibrium for the rod cannot be satisfied with forces whose magnitudes are not larger than the maxima of the forces of friction. We shall show that in this case the function $S + \Psi$ has no minimum. The function coincides in this case with the function

$$\Psi = -F\ddot{x}\cos\varphi - F(\ddot{y} + \varepsilon b)\sin\varphi + kN_1\sqrt{\ddot{x}^2 + \ddot{y}^2} + kN_2\sqrt{\ddot{x}^2 + (\ddot{y} + \varepsilon a)^2}$$

It is sufficient to show that the function Ψ can take on negative values in the given case. We note that it is sufficient for this purpose to find a point O_1, such that the moment of the force $\mathbf{F}_1$ about this point is larger than the sum of the moments of the frictional forces about this center.

Indeed, the work of all the forces for the possible displacement, which corresponds to an elementary rotation around O_1, will then be negative. Since in the given case the set of admissible accelerations and possible displacements coincide up to within a factor, the function Ψ can be made negative, for it is "the work of all forces on the admissible acceleration", taken with the opposite sign.

If $|bF\sin\phi| > kN_2a$ or $|(a-b)F\sin\phi| > kN_1a$, then Ψ will become negative when $\ddot{x} = \ddot{y} = 0$ or $\ddot{x} = \ddot{y} + \epsilon a = 0$, that is, for rotation around either m_1 or m_2.

If none of these inequalities is satisfied, then we apply to the rod two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ of magnitude kN_1 and kN_2, respectively. We apply them at the points m_1 and m_2 in such a way that their projections on the rod have the same sign, and that $F_{1y}a + (a-b)F\sin\phi = 0$, $F_{2y}a + bF\sin\phi = 0$.

If these forces are not parallel, then their perpendiculars will also intersect at some point O_1; if neither one of them is perpendicular to the rod, then O_1 will be the center of the resulting rotation for which Ψ will be negative.

Indeed, the equation of moments with respect to O_1 will be violated, since the point O_1 lies outside the x-axis, and if this equation were satisfied it would, together with the two preceding equations, form a complete system of equilibrium equations. But this is impossible by hypothesis.

If the forces are parallel then they will be oriented in the same direction, and if they are not perpendicular to the y-axis, then Ψ will be negative for displacements directed in the opposite direction of the forces.

In fact, the equation of equilibrium for the projections of these forces will be violated, otherwise all equations of equilibrium would be satisfied. If the forces are perpendicular to the x-axis, then $kN_1 + kN_2 = F\sin\phi$, and hence Ψ will become negative for a displacement along the force $\mathbf{F}$. If only one of the forces, say $\mathbf{F}_1$, is perpendicular to the

x-axis, and if its magnitude is kN_1, then the system of the three forces reduces to a nonvanishing resultant force F_3 directed along the x-axis in the direction of F_x.

Suppose that the angle between $\mathbf{F}_2$ and the x-axis is equal to α_2. If, without changing $\mathbf{F}_1$, one deflects it from the perpendicular to the x-axis through a small angle δ, then the center O_1, corresponding to these two forces, can be determined from the equations

$$x_1 \sin\delta - y_1 \cos\delta = 0, \qquad (x_1 - a)\cos\alpha_2 - y_1 \sin\alpha_2 = 0$$

The sum of the moments of all forces about O_1 will be

$$(F_3 + kN_1 \sin\delta)\, y_1 - kN_1 (1 - \cos\delta)\, x_1$$
$$= [(F_3 + kN_1 \sin\delta)\sin\delta - kN_1 (1 - \cos\delta)]\ \frac{a\cos\alpha_2}{\cos\delta\cos\alpha_2 - \sin\delta\sin\alpha_2}$$

For sufficiently small δ this sum can be made zero. Thus, $S + \Psi$ will surely not possess a minimum at the origin if it is impossible to satisfy the equilibrium equations with reactions (forces) whose magnitudes are not larger than kN_1 and kN_2.

Let us now assume that it is impossible to satisfy the second variant. For the second variant the function Π has the form

$$\Pi = (kN_2 + m_2\varepsilon^* a - F\sin\varphi)\,\ddot{y} - F\ddot{x}\cos\varphi + kN_1\sqrt{\ddot{x}^2 + \ddot{y}^2}$$

where ϵ^* is a solution of Equation (2.5).

It will take on only negative signs if, and only if,

$$(kN_2 + m_2\varepsilon^* a - F\sin\varphi)^2 + F^2\cos^2\varphi \leqslant k^2 N_1{}^2$$

It is not difficult to verify that this inequality is satisfied if, and only if, $R_1{}^2 \leqslant k^2 N_1{}^2$. Hence, the function $S + \Psi$ cannot have a minimum at the point $\ddot{x} = \ddot{y} = 0$, $\epsilon = \epsilon^*$ if the second variant is not satisfied. For the third variant the proof is analogous.

On the basis of the presented arguments, and because of property (a), we conclude that the motion agrees with the law of friction if, and only if $S + \Psi$ attains a minimum. Since in view of (c) such a minimum always exists, it follows that for arbitrary parameters there exists a motion which obeys the law of friction. If the rod is subjected to a pair of forces or to a force parallel to it, analogous results can be obtained.

Summing up, we find: there always exists a motion (or the state of rest) of a rod in accordance with the law of friction; this motion is unique and yields an isolated minimum of the function $S + \Psi$.

The determination of the motions under initial conditions which involve some zero relative velocites will be accomplished by the following scheme:

Suppose that at the initial instant $\mathbf{v}_1 = \ldots = \mathbf{v}_\tau = 0 (\tau \leqslant s)$.

Let us assume that $\dot{\mathbf{v}}_1 = \ldots = \dot{\mathbf{v}}_\nu = 0$, and one of the $\dot{\mathbf{v}}_{\nu+1}, \ldots, \dot{\mathbf{v}}_\tau$ is distinct from zero. Then assume that the reactions $\mathbf{R}_1, \ldots, \mathbf{R}_\nu$ at the points $p_1, \ldots, p_\nu$ are unknown, but lie in the cones of friction and are directed inward the body to which they are applied; the forces of friction at the points $p_{\nu+1}, \ldots, p_\tau$ will be assumed to be equal to $k_i N_i$ and to be directed oppositely to the vectors $\dot{\mathbf{v}}_{\nu+1}, \ldots, \dot{\mathbf{v}}_\tau$.

The choice of the frictional forces at the points $p_{\nu+1}, \ldots, p_\tau$ is dictated by the requirement that these, and only these, forces go over continuously into the forces which are directed against the relative velocity if the latter is not zero [1].

We note also that, since $\mathbf{v}_1, \ldots, \mathbf{v}_\tau$ are zero at the initial instant, we have at this time

$$(\dot{\mathbf{v}}_i)_{ix} = \dot{v}_{ix}, \qquad (\dot{\mathbf{v}}_i)_{iy} = \dot{v}_{iy} \qquad (i = 1, \ldots, \tau)$$

The equations of motion which are constructed in accordance with the indicated assumptions have the form

$$\left(\frac{\partial S}{\partial \ddot{q}_j}\right)^0 = Q_j + \sum_{i=\tau+1}^{s} - k_i N_i \frac{v_{ix}\alpha_{ij}^1 + v_{iy}\alpha_{ij}^2}{|\mathbf{v}_i|} + \alpha_{ij}^3 N_i +$$
$$+ \sum_{i=\nu+1}^{\tau} - k_i N_i \frac{\dot{v}_{ix}\alpha_{ij}^1 + \dot{v}_{iy}\alpha_{ij}^2}{|\dot{\mathbf{v}}_i|} + \alpha_{ij}^3 N_i + \sum_{i=1}^{\nu} R_{1i}\alpha_{ij}^1 + R_{2i}\alpha_{ij}^2 + R_{3i}\alpha_{ij}^3 \qquad (2.16)$$

Here the symbol $(\ \ )^0$ indicates that in the partial derivative we have set $\dot{q}_{n+1} = \ldots = \dot{q}_{n+k} = \dot{\mathbf{v}}_1 = \ldots = \dot{\mathbf{v}}_\nu = 0$. If after the imposition of these conditions only the quantities $\ddot{q}_1, \ldots, \ddot{q}_n, (n' < n)$ remain as independent generalized accelerations, and if Equations (2.1) can be satisfied by a set $\ddot{q}_1, \ldots, \ddot{q}_{n'}, N_i > 0, R_{3i}k \geqslant \sqrt{(R_{1i}^2 + R_{2i}^2)}$, then this motion does not contradict the law of friction. It it is, however, impossible to accomplish this, then one has to try a different but similar hypothesis, which consists in assuming that other relative accelerations are zero and the remaining ones are distinct from zero.

In the search for motions which will not contradict the law of friction it is necessary to make $C_\tau^1 + C_\tau^2 + \ldots + C_\tau^\tau = 2^\tau - 1$ trials, where C_τ^i is the number of combinations of τ elements with respect to i.

In both of the presented cases (1.6) and (2.16), the answer to the question on the existence and uniqueness of a system of generalized accelerations, satisfying Equations (1.6) and (2.16), remains open.

Painlevé has shown that there can occur cases when such equations have no solutions or when they have several solutions.

3. Below, we shall consider systems with initial conditions which correspond to several zero relative velocities and such that all $N_i > 0$, which appear in the equations for the determination of the motion, can be determined in a unique manner to within the determination of the generalized accelerations by the conditions

$$\ddot{q}_{n+1} = \ldots = \ddot{q}_{n+k} = 0$$

In view of the hypothesis made, one can consider as known the virtual work of the frictional forces at the points with zero relative velocities, because at these points the frictional forces are known in magnitude and direction.

Let us now assume that the relative velocities $\mathbf{v}_1, \ldots, \mathbf{v}_\tau$ vanish at the points of contact $p_1, \ldots, p_\tau$ at the initial moment, and suppose that among the quantities $v_{1x}, v_{1y}, \ldots, v_{\tau x}, v_{\tau y}$ there are σ ($\sigma \leqslant 2\tau$) independent variables $v_1, \ldots, v_\sigma$ and some v_{ix}, v_{iy} which can be expressed in terms of these v_i in the form

$$v_{ix} = \beta_{i1}{}^1 v_1 + \ldots + \beta_{i\sigma}{}^1 v_\sigma, \qquad v_{iy} = \beta_{i1}{}^2 v_1 + \ldots + \beta_{i\sigma}{}^2 v_\sigma \qquad (i = 1, \ldots, \tau) \quad (3.1)$$

For the indicated initial conditions $v_1 = \ldots = v_\sigma = 0$, we have

$$\dot{v}_{ix} = \beta_{i1}{}^1 \dot{v}_1 + \ldots + \beta_{i\sigma}{}^1 \dot{v}_\sigma, \qquad \dot{v}_{iy} = \beta_{i1}{}^2 \dot{v}_1 + \ldots + \beta_{i\sigma}{}^2 \dot{v}_\sigma \qquad (i = 1, 2, \ldots, \tau) \quad (3.2)$$

Furthermore, the possible displacements δx_i, δy_i can be expressed in terms of the possible displacements $v_1 \delta t, \ldots, v_\sigma \delta t$ by means of the formulas

$$\delta x_i = \beta_{i1}{}^1 v_1 \delta t + \ldots + \beta_{i\sigma}{}^1 v_\sigma \delta t, \qquad \delta y_i = \beta_{i1}{}^2 v_1 \delta t + \ldots + \beta_{i\sigma}{}^2 v_\sigma \delta t \qquad (3.3)$$

which are analogous to (3.1). We note that in Equations (3.1) and (3.3) we understand by $v_1, \ldots, v_\sigma$ some nonzero possible velocities distinct from the actual zero velocities, and the indicated representation of the possible displacement is valid because the relations do not contain the time explicitly.

Since v_{ix}, and v_{iy} are zero

$$(\dot{\mathbf{v}}_i)_{ix} = \dot{v}_{ix}, \qquad (\dot{\mathbf{v}}_i)_{iy} = \dot{v}_{ix}$$

Now let $v_1, \ldots, v_n$ be some system of nonholonomic variables which makes the system $v_1, \ldots, v_\sigma$ complete, and let $Q_1', \ldots, Q_n'$ be the generalized forces, which correspond to this system of variables, and are composed of the active forces and the known frictional forces at the points with nonzero relative velocities, while $S'(\dot{v}_1, \ldots, \dot{v}_n)$ is the energy of acceleration expressed in terms of new variables.

Let us assume that in consequence of the vanishing of all of the $\mathbf{v}_1, \ldots, \mathbf{v}_\tau$, all of the $v_1, \ldots, v_n$ also vanish.

The hypothesis that $\dot{v}_1, \ldots, \dot{v}_n$ are all zero is in accord with the law of friction if the equations

$$Q_j' + \sum_{i=1}^{\tau} R_{1i}\beta_{ij}^1 + R_{2i}\beta_{ij}^2 = 0 \qquad (j = 1, 2, \ldots, n) \tag{3.4}$$

can be satisfied simultaneously with the inequalities

$$R_{1i}^2 + R_{2i}^2 \leqslant k_i^2 N_i^2 \tag{3.5}$$

where the N_i are known by hypothesis.

We shall show that if (3.4) and (3.5) are satisfied, then the function

$$S' + \Psi = S' - \sum_{i=1}^{n} Q_i' \dot{v}_i + \sum_{i=1}^{\tau} k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}$$

has an isolated minimum at the origin. For this purpose it is necessary and sufficient that Ψ can take on only non-negative values in a neighborhood of the origin.

The sufficiency is obvious, since under the given initial conditions S' is a positive-definite quadratic form in the variables $\dot{v}_1, \ldots, \dot{v}_n$. For the proof of the necessity, let us assume the opposite, that is, we shall suppose that $\Psi(\dot{v}_1^\circ, \ldots, \dot{v}_n^\circ) = -\gamma < 0$. Since Ψ is a homogeneous function of the first order, it follows that $\Psi(t\dot{v}_1^\circ, \ldots, t\dot{v}_n^\circ) = -t\gamma$. One can always find an $R^2 > 0$ such that $S' \leqslant R^2(\dot{v}_1^2 + \ldots + \dot{v}_n^2)$. Therefore, $S' + \Psi < R^2(\dot{v}_1^2 + \ldots + \dot{v}_n^2) + \Psi$, and at the points tv_i° we shall have $S' + \Psi \leqslant R^2 t^2(\dot{v}_1^{\circ 2} + \ldots + \dot{v}_n^{\circ 2}) - t\gamma$. It is clear that when $t > 0$ is sufficiently small than $S' + \Psi$ can become negative.

If Equations (3.4) are satisfied under the conditions (3.5), then Ψ can be expressed in the form

$$\Psi = \sum_{i=1}^{\tau} R_{1i}\dot{v}_{ix} + R_{2i}\dot{v}_{iy} + k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}$$

It is clear that Ψ will be non-negative if each term of this sum is non-negative. The condition for its negativeness is

$$k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2} \geqslant - R_{1i}\dot{v}_{ix} - R_{2i}\dot{v}_{iy}$$

After squaring both sides of this inequality we obtain

$$k_i^2 N_i^2 (\dot{v}_{ix}^2 + \dot{v}_{iy}^2) \geqslant R_{1i}^2 \dot{v}_{ix}^2 + R_{2i}^2 \dot{v}_{iy}^2 + 2R_{1i}R_{2i}\dot{v}_{ix}\dot{v}_{iy}$$

In accordance with Sylvester's criterion, the last inequality will be satisfied if

$$\begin{vmatrix} k_i^2 N_i^2 - R_{1i}^2 & R_{1i}R_{2i} \\ R_{1i}R_{2i} & k_i^2 N_i^2 - R_{2i}^2 \end{vmatrix} \geqslant 0$$

This condition is obviously equivalent to

$$k_i^2 N_i^2 \geqslant R_{1i}^2 + R_{2i}^2$$

which is (3.5), as was to be proved.

The proof of the converse assertion, that is, the solvability of (3.4) under the conditions (3.5) when Ψ is non-negative, we have not been able to accomplish. One can, however, prove that if Ψ is non-negative in the neighborhood of the origin, then the assumption that at least some accelerations, say $\dot{v}_1$ and $\dot{v}_2$, are distinct from zero leads to a contradiction. In fact, for the determination of these accelerations we obtain the equations

$$\frac{\partial S'^{\circ}}{\partial \dot{v}_1} = Q_1' - \sum k_i N_i \frac{\dot{v}_{ix}{}^{\circ}\beta_{i1}{}^1 + \dot{v}_{iy}{}^{\circ}\beta_{i1}{}^2}{\sqrt{\dot{v}_{ix}{}^{\circ 2} + \dot{v}_{iy}{}^{\circ 2}}}$$

$$\frac{\partial S'^{\circ}}{\partial \dot{v}_2} = Q_2' - \sum k_i N_i \frac{\dot{v}_{ix}{}^{\circ}\beta_{i2}{}^1 + \dot{v}_{iy}{}^{\circ}\beta_{i2}{}^2}{\sqrt{\dot{v}_{ix}{}^{\circ 2} + \dot{v}_{iy}{}^{\circ 2}}}$$

where $S'^{\circ} = S'(\dot{v}_1, \dot{v}_2, 0, 0, \ldots, 0)$, and $\dot{v}_{ix}{}^{\circ}$ and $\dot{v}_{iy}{}^{\circ}$ are $\dot{v}_{ix}$ and $\dot{v}_{iy}$, and where we have set $\dot{v}_3 = \ldots = \dot{v}_{\sigma} = 0$. The summation is performed over all i that correspond to the vanishing $\dot{\mathbf{v}}_i$. Multiplying the equations by $\dot{v}_1$ and $\dot{v}_2$ respectively, adding the results, and taking into account that under the initial conditions $v_1 = \ldots = v_{\sigma} = 0$ S'° is a positive-definite quadratic form in $\dot{v}_1, \ldots, \dot{v}_{\sigma}$, we obtain

$$2S'^{\circ} = Q_1'\dot{v}_1 + Q_2'\dot{v}_2 - \sum k_i N_i \sqrt{\dot{v}_{ix}{}^{\circ 2} + \dot{v}_{iy}{}^{\circ 2}}$$

Since $S'^{\circ} > 0$, we have a contradiction which shows that either the presence of a minimum of $S' + \Psi$ at the origin guarantees that $\dot{v}_1 = \ldots = \dot{v}_{\sigma} = 0$, or the system cannot be in the state of rest nor in any kind of motion that would obey the law of friction. In this case we shall say that the law of friction is contradictory.

4. Let us consider the general case when at the initial instant $v_1, \ldots, v_\sigma$ are zero, $\sigma < n$.

Suppose that the assumptions that $\dot{\mathbf{v}}_1, \ldots, \dot{\mathbf{v}}_\nu$ are zero and that none of the $\dot{\mathbf{v}}_{\nu+1}, \ldots, \dot{\mathbf{v}}_n$ is zero lead to the single proposition that $\dot{v}_1 = \ldots = \dot{v}_\mu = 0 (\mu \leqslant 2\nu)$. This will be the case if the first $v_1, \ldots, v_\mu$ are selected from the velocities $v_{ix}, v_{iy}, \ldots, v_{\nu x}, v_{\nu y}$ so that they are independent.

For the determination of the $\dot{v}_{\mu+1}, \ldots, \dot{v}_n$ one can use the equations

$$\frac{\partial S^*}{\partial \dot{v}_j} = Q_j' - \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix}\beta_{ij}^1 + \dot{v}_{iy}\beta_{ij}^2}{\sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}} \qquad (j = \mu + 1, \ldots, n)$$

where S^* is S, and where we have set $\dot{v}_1 = \ldots = \dot{v}_\mu = 0$. It is not difficult to see that the preceding equations can be written in the form

$$\frac{\partial}{\partial \dot{v}_j}\Big(S^* - \sum_{i=\mu+1}^{n} Q_i'\dot{v}_i + \sum_{=\nu+1}^{\tau} k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2} \Big) = \frac{\partial}{\partial \dot{v}_i}(S^* + \Psi^*) \qquad (4.1)$$

$$(j = \mu + 1, \ldots, n)$$

If $\dot{v}_{\mu+1}, \ldots, \dot{v}_n$ are a solution of these equations, then the motion

$$\dot{v}_1 = \ldots = \dot{v}_\mu = 0, \qquad \dot{v}_{\mu+1}^*, \ldots, \dot{v}_n^* \qquad (4.2)$$

will agree with the law of friction, provided that the equations

$$\left(\frac{\partial S}{\partial \dot{v}_j}\right)^* = Q_j' - \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix}^*\beta_{ij}^1 + \dot{v}_{iy}^*\beta_{ij}^2}{\sqrt{\dot{v}_{ix}^{*2} + \dot{v}_{iy}^{*2}}} + \sum_{i=1}^{\nu} R_{1i}\beta_{ij}^1 + R_{2i}\beta_{ij}^2$$

$$(j = 1, \ldots, \mu) \qquad (4.3)$$

in which $\dot{v}_{ix}^*$, $\dot{v}_{iy}^* (\partial S/\partial \dot{v}_j)^*$ indicate that the values (4.2) have been substituted for $\dot{v}_1, \ldots, \dot{v}_n$, are satisfied by the reactions R_{1i}, R_{2i} lying in the cone of friction

$$R_{1i}^2 + R_{2i}^2 \leqslant k_i^2 N_i^2 \qquad (i = 1, \ldots, \nu) \qquad (4.4)$$

We shall show that the solution (4.2), which agrees with the law of friction, yields an isolated minimum of the function

$$S' + \Psi = S' - \sum_{i=1}^{n} Q_i' \dot{v}_i + \sum_{i=1}^{\tau} k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}$$

In fact, $\delta(S + \Psi)$, the variation of the function $S + \Psi$ in the neighborhood (4.2) will in consequence of (4.1) have the form

$$\delta(S' + \Psi) = \sum_{j=1}^{\mu} \left[\left(\frac{\partial S}{\partial \dot{v}_j} \right)^* - Q_j' + \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix}^* \beta_{ij}^1 + \dot{v}_{ix}^* \beta_{ij}^2}{\sqrt{\dot{v}_{ix}^{*2} + \dot{v}_{iy}^{*2}}} \right] \dot{v}_j + \\ + \sum_{i=1}^{\nu} k_i N_i \sqrt{\dot{v}_{ix}^1 + \dot{v}_{iy}^2} + \delta^2 S' + \delta^2 \Psi^* + \lambda$$

where λ stands for terms of order higher than the second, and $\delta^2 S'$ and $\delta^2 \Psi^*$ represent all second-order terms in the expansion of the functions S' and Ψ^*.

Since

$$\delta^2 \Psi^* = \frac{1}{2} \delta \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix} \delta \dot{v}_{ix} + \dot{v}_{iy} \delta \dot{v}_{iy}}{\sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}} = \frac{1}{2} \sum_{i=\nu+1}^{\tau} \frac{k_i N_i (\dot{v}_{iy}^* \delta \dot{v}_{ix} - \dot{v}_{ix}^* \delta \dot{v}_{iy})^2}{\sqrt{(\dot{v}_{ix}^{*2} + \dot{v}_{iy}^{*2})^3}}$$

is an everywhere positive function of $\delta \dot{v}_{ix}$, $\delta \dot{v}_{iy}$, while $\delta^2 S$ is a positive-definite function of $\delta \dot{v}_i$, it is not difficult to show by a method analogous to the one used in the preceding section, that a necessary and sufficient condition for the positiveness of $\delta(S + \Psi)$ is the non-negativeness of the following homogeneous function of the first degree:

$$\Pi = \sum_{j=1}^{\mu} \left[\left(\frac{\partial S'}{\partial \dot{v}_j} \right)^* - Q_j' + \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix}^* \beta_{ij}^1 + \dot{v}_{iy}^* \beta_{ij}^2}{\sqrt{\dot{v}_{ix}^{*2} + \dot{v}_{iy}^{*2}}} \right] \dot{v}_j + \\ + \sum_{i=1}^{\nu} k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2} \geqslant 0$$

This last inequality will be satisfied if Equations (4.3) can be satisfied by reactions lying in the cone of friction. The proof of the last statement can be carried out in the same way as in the case of equilibrium, with the only difference that the generalized forces in Formula (3.4) have to be replaced by the quantities which are in the square brackets.

It is not difficult to see that the non-negativeness of $\delta^2\Psi^*$ guarantees the positive definiteness of $\delta^2 S^* + \delta^2 \Psi^*$ as a function of the $\dot{v}_{\mu+1}$, $\ldots$, $\dot{v}_n$. This in turn shows that the solution of Equations (4.1) is unique in any region where the $v_{\nu+1}, \ldots, v_\tau$ do not vanish. In fact, the Jacobian of the left-hand sides of Equations (4.1) exists for all $\dot{v}_{\mu+1}, \ldots, \dot{v}_n$ that do not make $v_{\nu+1}, \ldots, v_\tau$ equal to zero, and it coincides with the discriminant of the quadratic form $\delta^2 S^* + \delta^2 \Psi^*$, which is positive wherever it exists.

If it is impossible to solve Equations (4.3) under the restrictions (4.4), but the last inequality is still valid for arbitrary $\dot{v}_1, \ldots, \dot{v}_\mu$, then the law of static friction is contradictory.

Indeed, by adding to all the forces acting on the system the forces of inertia

$$-\left(\frac{\partial S'}{\partial \dot{v}_j}\right)^* \qquad (j = 1,2,\ldots,\mu); \; -\frac{\partial S^*}{\partial \dot{v}_j} \qquad (j = \mu+1,\ldots,n)$$

we consider our system in its initial condition $v_1, \ldots, v_n = 0$ subjected to the active forces

$$Q_j^\circ = -\left(\frac{\partial S}{\partial \dot{v}_j}\right)^* + Q_j' - \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix}^* \beta_{ij}^1 + \dot{v}_{iy}^* \beta_{ij}^2}{\sqrt{\dot{v}_{ix}^{*2} + \dot{v}_{iy}^{*2}}} \qquad (j = 1,2,\ldots,\mu)$$

$$Q_j^\circ = -\frac{\partial S^*}{\partial \dot{v}_j} + Q_j' - \sum_{i=\nu+1}^{\tau} k_i N_i \frac{\dot{v}_{ix}^* \beta_{ij}^1 + \dot{v}_{iy}\, \beta_{ij}^2}{\sqrt{\dot{v}_{ix}^{*2} + \dot{v}_{iy}^{*2}}} = 0 \quad (j = \mu+1,\ldots,n)$$

and the unknown forces of friction at the points $p_1, \ldots, p_\nu$. We thus obtain a substitute system which differs from the original one by the initial conditions, and also by the fact that the forces of inertia and friction which were computed under the assumptions that the $\dot{\mathbf{v}}_1, \ldots, \dot{\mathbf{v}}_\nu$ were zero, are replaced by active forces. This substitute system cannot have any "motion", due to the active forces Q_j°, that can agree with the law of friction. In fact, for the determination of any nonzero $\dot{v}_1', \ldots, \dot{v}_n'$ for the substitute system, we have the equations

$$\frac{\partial S}{\partial \dot{v}_j} = Q_j^\circ - \sum_{i=1}^{\nu} k_i N_i \frac{\dot{v}_{ix}\beta_{ij}^1 + \dot{v}_{iy}\beta_{ij}^2}{\sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}} \qquad (j = 1,2,\ldots,\mu)$$

$$\frac{\partial S}{\partial \dot{v}_j} = 0 \qquad (j = \nu+1,\ldots,n)$$

Multiplying these equations by $\dot{v}_j'$, adding the results, and taking into account the initial conditions ($v_1 = \ldots = v_n = 0$), we find that their solutions $\dot{v}_1', \ldots, \dot{v}_n'$ satisfy the condition

$$2S = -\Pi > 0$$

Therefore, the assumption that at least one of the $\dot{v}_1', \ldots, \dot{v}_n'$ is different from zero is contradictory to the assumption that $\Pi \geqslant 0$.

Finally, we arrive at the conclusion that if one stops the moving system and adds to the active forces the forces of inertia and the known forces of friction, then one will necessarily have equilibrium ($\dot{v}_1 = \ldots = \dot{v}_n = 0$) for this substitute system when $\Pi \geqslant 0$, in all cases when the law of friction of the substitute system is not contradictory. It is not difficult to see the similarity between this statement and D'Alembert's principle, for it differs from the latter only in the respect that we can draw conclusions about the equilibrium only after we have introduced the assumption $\dot{\mathbf{v}}_1 = \ldots = \dot{\mathbf{v}}_\nu = 0$ and have obtained from it the frictional forces and the forces of inertia.

Thus, two conclusions can be made if the law of friction is not contradictory for the substitute system.

1) From the non-negativeness of the function Π it follows that the solution (4.2) is in accord with the law of friction.

2) The system which is in the state of rest under the influence of some active forces, frictional forces and forces of inertia, cannot have accelerations which correspond to these forces of inertia if it is under the influence of the same active forces and frictional forces.

The last assertion does not seem probable, and so we shall dwell upon the first.

We shall explain what has been said with an example. Let us consider a rigid triangle with vertices A, B, C, which is pressed at these vertices against a rough immovable plane by forces $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ normal to the plane.

To the vertex A there is hinged an inertialess rod which has at the end D a mass m that is pressed against the plane by the force $\mathbf{N}_4$. Suppose that the rod is subjected to a force $\mathbf{F}$, lying in the plane at a distance b from the point A and forming an angle α with the rod.

If $|bF \sin \alpha| > kN_4 a$, where $a = AD$, then from the equation

$$bF \sin \alpha = (kN_4 + m\varepsilon a)a$$

one can find the force of inertia $m\epsilon a$ and the frictional force kN_4, which are directed perpendicularly to the rod in the opposite direction of the acceleration of its end. These are the inertia force and the frictional force found under the hypothesis that the accelerations of the vertices of the triangle are zero. If one now applies an active force $kN_4 + m\epsilon a$ at the point D to the rod, and if one considers the substitute system, then, when $\Pi > 0$, the triangle and rod will necessarily be in equilibrium if the law of friction is not contradictory. The first proposition confirms the admissiblity of this motion, while the second proposition denies it.

We note that it is always possible to find an $R^2 > 0$ such that on the sphere $\dot{v}_1^{\,2} + \ldots + \dot{v}_n^{\,2} = R^2$ the values of the function $S + \Psi$ will be positive.

Since $S + \Psi$ is continuous in the closed region

$$\dot{v}_1^2 + \ldots + \dot{v}_n^2 \leqslant R^2$$

and is equal to zero at the origin, it follows from a known theorem in analysis that this function attains a minimum within this region, and that this minimum, in accordance with the structure of the variation of $S + \Psi$, will be an isolated minimum.

5. As was pointed out by Appell [2], the function $S' - Q_1'\dot{v}_1 - \ldots - Q_n'\dot{v}_n$ differs from the value of the Gauss constraint only by a constant. Therefore, it is natural to give a general formulation of a principle for systems with friction that is analogous to Gauss's principle, and which will have the advantage that it excludes the cases of intrinsic paradoxes of the law of friction whenever they arise.

Since, however, $S + \Psi$ for real motion can take on also negative values, we shall not use the term constraint.

The quantity

$$-\Psi = +\sum_{i=1}^{n} Q_i'\dot{v}_i - \sum_{i=1}^{\tau} k_i N_i \sqrt{\dot{v}_{ix}^{\,2} + \dot{v}_{iy}^2}$$

can be called the work of all the forces applied to the system with a virtual acceleration. By a virtual acceleration we shall mean, following Gauss, an acceleration which agrees with the conditions imposed on the system and in which $q_{n+1} = \ldots = q_{n+k} = 0$.

Formulation of the principle. If in a system with contacts one knows all maxima of the frictional forces, then the actual motion which agrees with the law of friction will differ from all neighboring conceivable motions by the fact that for a real motion the difference between the energy of acceleration and the work of all forces on the real acceleration will be smaller than that same difference is for any virtual acceleration that is near a real one; furthermore, there will always exist at least one motion which satisfies this condition.

6. Example. Let us consider a solid body which rests with a flat part of its surface upon a rough plane. Following Zhukovskii [3], we shall assume that the normal pressure N_i is known at every element of contact d.

Zhukovskii has shown the following:

a) If the body is rotated about any center O_i, the forces of friction can be reduced to some force $\mathbf{F}_i$ in all cases except for a rotation of a unique point O, the pole of friction, when the frictional forces reduce to a couple.

b) Along any straight line in the plane there acts a unique force $\mathbf{F}_i$.

c) The moment of the force $\mathbf{F}_i$ about O_i is not less than the moment of any other frictional force about the same center.

d) The frictional force which corresponds to any translational motion is unique and passes through the center of normal pressure.

Utilizing these properties, Zhukovskii derives Theorem IV.

Theorem IV. A necessary and sufficient condition for the equilibrium of a solid body resting on a fixed plane is that the force **P** acting on the body along a line in the fixed plane be not greater than the force of friction in the direction of this line. If, however, a couple lying in the fixed plane acts on the body, then a necessary and sufficient condition for the body to be in equilibrium is that the moment of this couple be not greater than the moment of the couple of friction obtained by revolving the body about its pole of friction.

Zhukovskii arrived at this theorem after he had established that the equations of motion, obtained from the theorem of angular momentum applied either to the centers O_i or to O, or from the motion of the center of mass in the projection on the direction **P**, could have no solution. It is not difficult to verify that the conditions of Zhukovskii's theorem coincide with the necessary and sufficient conditions for the non-negativeness of the function Ψ for the given problem. In fact, from the condition of the theorem it can be seen, that as soon as these conditions are satisfied, the virtual work of the force **P** and of the frictional forces will always be nonpositive. From the proportionality of the fields of virtual accelerations and possible displacements, one can deduce the non-negativeness of Ψ. If the ratio of the force **P** to the friction force $\mathbf{F}_1$, directed along the line of action of **P**, is equal to $|\mathbf{P}|/|\mathbf{F}_i| = \lambda \leqslant 1$, then if one applies at all elements of contact forces directed in the same way as the force which balances $\mathbf{F}_i$, and have magnitudes $\lambda_k N_i d\sigma$, where k is the coefficient of friction, then we obtain the force $-\lambda\mathbf{F} = \mathbf{P}$, directed along the line of **P**, since the equations of the lines of action of the balancing forces are homogeneous in the components of the component forces. The proof is analogous for the case of a couple.

If the conditions of the theorem are violated, then it is impossible to make an analogous selection of forces. Indeed, in the case when we have a couple and a force applied at the center of pressure, the equation of moments relative to the pole of friction or the equation of equilibrium for the projection on **P**, will surely be violated. In the general case, the moment of arbitrary forces, applied at the points of contact and of absolute value less than the forces of friction, will be less in magnitude than the moment of $\mathbf{F}_i$ about the point O_i. In fact, it is equal to

$$\iint_D kN_i r_i d\sigma.$$

where r_i is the distance of the element d from the center O_i. Therefore, a change in direction or a decrease of the moduli of the forces of friction can cause only a decrease of their moment about O_i.

If the conditions of the theorem are violated, then one can use the following equations for the determination of the initial accelerations:

$$m\ddot{x}_c = P_x - \iint_D kN_i \frac{(\ddot{x}_c - r\varepsilon \sin\varphi)\, rdrd\varphi}{\sqrt{(\ddot{x}_c - r\varepsilon\sin\varphi)^2 + (\ddot{y}_c + r\varepsilon\cos\varphi)^2}}$$

$$m\ddot{y}_c = P_y - \iint_D kN_i \frac{(\ddot{y}_c + r\varepsilon\sin\varphi)\, rdrd\varphi}{\sqrt{(\ddot{x}_s - r\varepsilon\sin\varphi)^2 + (\ddot{y}_c + r\varepsilon\cos\varphi)^2}} \qquad (6.1)$$

$$I_c\varepsilon = Ph_c - \iint_D kN_i \frac{[-(\ddot{x}_c - r\varepsilon\sin\varphi)\, r\sin\varphi + (\ddot{y}_c + r\varepsilon\cos\varphi)\, r\cos\varphi]\, rdrd\varphi}{\sqrt{(\ddot{x}_c - r\varepsilon\sin\varphi)^2 + (\ddot{y}_c + r\varepsilon\cos\varphi)^2}}$$

where $\ddot{x}_c$ and $\ddot{y}_c$ are the accelerations of the center of mass of the body, J_c its central moment of inertia, ϵ is the angular velocity, h_r the distance of the force **P** from the center of the mass, and the region D is the area of contact.

The solution of the indicated equations is given by the minimum of the function

$$S + \Psi = \frac{m}{2}(\ddot{x}_c^2 + \ddot{y}_c^2) + \frac{J_c\varepsilon^2}{2} - P_x\ddot{x}_c - P_y\ddot{y}_c - Ph_c\varepsilon + \iint_D k_iN_i \sqrt{(\ddot{x}_c - r\varepsilon\sin\varphi)^2 + (\ddot{y}_c + r\varepsilon\cos\varphi)^2}$$

This function possesses second-order partial differential derivatives at all points except at $\ddot{x}_c = \ddot{y}_c = \epsilon = 0$. It must have a minimum at some point distinct from this one, because the conditions of the theorem of Zhukovskii are violated. Hence Equations (6.1) have a solution, and this solution is unique, since any two points of the space of accelerations can be connected by a continuous curve which passes through the origin. Thus we have the following result:

1) The necessary and sufficient conditions for the Zhukovskii equilibrium coincide with the necessary and sufficient conditions obtained by the direct application of the law of friction.

2) The law of static friction is not contradictory in this case.

3) The application of the law of friction, when the condition of equilibrium is violated, leads to a unique solution which agrees with the law of friction.

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